ON THE AERODYNAMICS OF FIN-STABILIZED AXISYMMETRIC BODIES AT ANGLE OF ATTACK

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Starting from rather broad assumptions concerning the dependence of aerodynamic parameters on the orientation of the free-stream velocity relative to the body axes, the present paper studies the implications of various symmetries of the body on these parameters. Expressions for the dependence of the aerodynamic parameters on the polar angle θ of the body cylindrical coordinate system are developed. The aerodynamic forces acting on an axisymmetric body with control surfaces are expressed as functions of the orientation of the body, and also of the number of fins in the absence of mutual interference between the fins.

1. Let us study the dependence of an aerodynamic parameter F on the direction of the free-stream velocity **V** relative to the body. We shall denote by e_x, e_y, e_z the projections of the unit vector $\mathbf{e} = \mathbf{V}/V$ on the rectangular coordinates x, y, z associated with the body. When $e_x \geqslant 0$, the direction of **V** is uniquely determined by the values of e_y, $e_z (e_x = (1 - e_y^2 - e_z^2)^{1/2})$. Hence, we shall study the dependence $F = F(e_y, e_z)$. The parameter F may refer to fixed x, y, z points (for instance, to points on the surface of the body) or to points with coordinates dependent on e_y and e_z (for instance to points on the shock front*). F can also represent a quantity characteristic of the overall flow (for instance, any component of the aerodynamic forces or moments).

We assume that, when $(e_y^2 + e_z^2)^{1/2} \leqslant M_*$ $(M > 0)$, the function $F(e_y, e_z)$ can be expressed with sufficient accuracy by Taylor series, terminated with the terms of nth power:

* If we should refer the value F on a shock front to points fixed in the x, y, z system, then the function $F(e_y, e_z)$ for such points cannot be expressed in the assumed form (1.1).

$$F(e_y, e_z) = \sum_{k=0}^{n} \sum_{i=0}^{k} f_{ik} e_y{}^i e_z{}^{k-i} \qquad (1.1)$$

All the subjequent developments are based on this assumption. Let us change variables in (1.1), as indicated in Fig. 1:

$$e_y = \sin\alpha\cos\beta, \qquad e_z = \sin\alpha\sin\beta \quad (1.2)$$

Here α represents the angle between the x-axis and the direction of **V** (the angle of attack), β the angle between the plane $z = 0$ and the plane formed by **V** and the x-axis, called the plane of the angle of attack hereafter. Substituting (1.2) into (1.1), we find the dependence $F(\alpha, \beta)$:

$$F(\alpha, \beta) = \sum_{k=0}^{n} F_k(\beta) \sin^k \alpha \qquad (1.3)$$

$$F_k(\beta) = \sum_{i=0}^{k} f_{ik} \cos^i \beta \sin^{k-i} \beta \qquad (1.4)$$

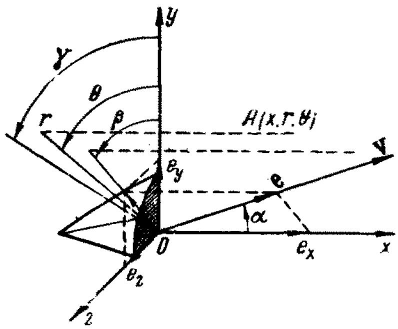

Fig. 1.

According to our basic assumption, the equality (1.3) is valid for all values of β and for $0 \leqslant \sin\alpha \leqslant M$.

With the aid of identities between powers of trigonometric functions and functions of multiple angles [1], it is possible to express the right-hand side of (1.4) in the form

$$F_k(\beta) = \sum_{0 \leqslant j \leqslant k/2} [f^c_{k-2j,\,k} \cos(k-2j)\beta + f^s_{k-2j,\,k} \sin(k-2j)\beta] \qquad (1.5)$$

(The coefficients of (1.5) and (1.4) are related one to one.)

Let us introduce a cylindrical coordinate system x, r, θ, having the x-axis in common with the system x, y, z. The parameter F will thus refer to the point $A(x, r, \theta)$. Furthermore, γ will indicate an angle by which the body is rotated around the x-axis from some original orientation (Fig. 1). Let us examine the dependence of F not only on α, β, but also on the angles γ, θ with the assumption that for at least one pair of values γ, θ, Equation (1.3) is valid. Then there exists a function $M(\gamma, \theta) > 0$, not identically zero, such that for $0 \leqslant \sin\alpha \leqslant M(\gamma, \theta)$ and for all values β, γ, θ, this dependence F can be represented in a form analogous to (1.3) and (1.5):

$$F(\alpha, \beta, \gamma, \theta) = \sum_{k=0}^{n} F_k(\beta, \gamma, \theta) \sin^k \alpha \tag{1.6}$$

$$F_k(\beta, \gamma, \theta) = \sum_{0 \leqslant j \leqslant k/2} [f^c_{k-2j, k}(\gamma, \theta) \cos(k-2j)\beta + f^s_{k-2j, k}(\gamma, \theta) \sin(k-2j)\beta] \tag{1.7}$$

It is clear that simultaneous increments in the angles β, γ, θ by an arbitrary angle ϕ do not change the quantity F. (If F is a projection of a vector, we suppose that the projection is effected onto the direction of x, or of r, or of θ:

$$F(\alpha, \beta, \gamma, \theta) = F(\alpha, \beta+\varphi, \gamma+\varphi, \theta+\varphi) \tag{1.8}$$

It follows

$$F_k(\beta, \gamma, \theta) = F_k(\beta+\varphi, \gamma+\varphi, \theta+\varphi) \tag{1.9}$$

It is possible to choose the function $M(\gamma, \theta)$ so that

$$M(\gamma, \theta) = M(\gamma+\varphi, \theta+\varphi) \tag{1.10}$$

Then, in accordance with (1.7) and (1.9)

$$F_k(\beta, \gamma, \theta) = \sum_{0 \leqslant j \leqslant k/2} [f^c_{k-2j, k}(\gamma+\varphi, \theta+\varphi) \cos(k-2j)(\beta+\varphi) + f^s_{k-2j, k}(\gamma+\varphi, \theta+\varphi) \sin(k-2j)(\beta+\varphi)] \tag{1.11}$$

Later cases will occur in which F by its nature does not depend on γ or θ. In these special cases the arguments $\gamma(\gamma + \phi)$ or $\theta(\theta + \phi)$ will be absent from Formulas (1.6) to (1.11). The function $M(\gamma, \theta) > 0$ will have a constant magnitude larger than zero, as follows from (1.10).

2. Let the x-axis coincide with the axis of the body of revolution. In that case the aerodynamic parameters will not depend on the angle of rotation around the x-axis, and (1.6) and (1.11) can be written

$$F(\alpha, \beta, \theta) = \sum_{k=0}^{n} F_k(\beta, \theta) \sin^k \alpha \tag{2.1}$$

$$F_k(\beta, \theta) = \sum_{0 \leqslant j \leqslant k/2} [f^c_{k-2j, k}(\theta+\varphi) \cos(k-2j)(\beta+\varphi) + f^s_{k-2j, k}(\theta+\varphi) \sin(k-2j)(\beta+\varphi)] \tag{2.2}$$

By virtue of the arbitrary choice of ϕ, let us set $\phi = -\theta$ in (2.1) and (2.2). We shall take the plane $\beta = 0$ as that of the angle of attack, and instead of $F(\alpha, 0, \theta)$, $f^c_{k-2j,k}(0)$, $f^s_{k-2j,k}(0)$ we shall write $F(\alpha, \theta)$, $f^c_{k-2j,k}$, $f^s_{k-2j,k}$. Then we obtain the form of F as dependent on θ:

$$F(\alpha, \theta) = \sum_{k=0}^{n} \sin^k \alpha \sum_{0 \leqslant j \leqslant k/2} [f_{k-2j,k}^{c} \cos (k-2j)\theta - f_{k-2j,k}^{s} \sin (k-2j)\theta] \quad (2.3)$$

The flow field around the axisymmetric body is symmetric with respect to the plane of the angle of attack. If the quantity F does not depend on the sense of the angle (i.e. on the directionality of the z-axis), then this symmetry implies $F(\alpha, \theta) = F(\alpha, -\theta)$. If the aerodynamic parameter changes sign with the sense of θ (and z-direction), then $F(\alpha, \theta) = -F(\alpha, -\theta)$.

From (2.3) it follows that in the first case $f^s_{k-2j,k} = 0$, and

$$F(\alpha, \theta) = \sum_{k=0}^{n} \sin^k \alpha \sum_{0 \leqslant j \leqslant k/2} f_{k-2j,k}^{c} \cos (k-2j)\theta \quad (2.4)$$

In the second case, $f^c_{k-2j,k} = 0$ and

$$F(\alpha, \theta) = -\sum_{k=0}^{n} \sin^k \alpha \sum_{0 \leqslant j \leqslant k/2} f_{k-2j,k}^{s} \sin (k-2j)\theta \quad (2.5)$$

Formulas (2.4) and (2.5) exhibit clearly the dependence of the aerodynamic parameters on the coordinate θ of the point at which the parameters are to be evaluated. If, in a numerical computation of the flow around an axisymmetric body at angle of attack, the aerodynamic parameters are sought in the form (2.4) or (2.5), then the three-variable problem reduces to a two-variable problem of determination of the coefficients $f^c_{k-2j,k}$ or $f^s_{k-2j,k}$, which depend only on x and r.

In many papers (for instance [2]), quantities of order α^2 are neglected in the solution of the flow around an axisymmetric body at a small angle of attack α. Then the relations of the form

$$F(\alpha, \theta) = f_{00}^{c} + f_{11}^{c} \alpha \cos \theta \quad \text{or} \quad F(\alpha, \theta) = -f_{11}^{s} \alpha \sin \theta \quad (2.6)$$

are utilized. The validity of (2.6) is generally justified on the bases that they satisfy approximately the aerodynamic equations and the boundary conditions. It is easy to see that the equalities (2.6) follow from (2.4) and (2.5), if in these the quantities of order α^2 are neglected.

3. The system of forces and moments acting on the body due to the flow can be represented by resultant vectors **R** of the aerodynamic forces and **M** of the aerodynamic moments. Henceforth, **F** will be understood to stand either for **R** or **M**. Let us designate the projections of **F** on x, z, z-axes by F_x, F_y, F_z and its projection on the ray $r(x = \text{const}, \theta = \text{const})$ by F_r. The aerodynamic parameters F_x, F_y, F_z characterize the whole flow field and do not depend on θ. The values F_y, F_z are related to F_r:

$$F_r(\alpha, \beta, \gamma, \theta) = F_y(\alpha, \beta, \gamma) \cos\theta + F_z(\alpha, \beta, \gamma) \sin\theta \tag{3.1}$$

$$F_y(\alpha, \beta, \gamma) = F_r(\alpha, \beta, \gamma, 0), \qquad F_z(\alpha, \beta, \gamma) = F_r(\alpha, \beta, \gamma, {}^1\!/_2\pi) \tag{3.2}$$

Let us apply Formula (1.8) to F_x and F_r, setting $\phi = -\gamma$:

$$F_x(\alpha, \beta, \gamma) = F_x(\alpha, \beta - \gamma, 0), \qquad F_r(\alpha, \beta, \gamma, \theta) = F_r(\alpha, \beta - \gamma, 0, \theta - \gamma) \tag{3.3}$$

The right-hand side of (3.3) is rewritten with the aid of (3.1):

$$F_r(\alpha, \beta, \gamma, \theta) = F_y(\alpha, \beta - \gamma, 0)\cos(\theta - \gamma) + F_z(\alpha, \beta - \gamma, 0)\sin(\theta - \gamma) \tag{3.4}$$

Let us develop the expressions for F_x, F_y, F_z on the right of (3.3) and (3.4) in accordance with (1.6) and (1.7) and let us specify the orientation of the plane of angle of attack by setting $\beta = 0$:

$$F_x(\alpha, \gamma) = \sum_{k=0}^{n} \sin^k\alpha \sum_{0 \leqslant j \leqslant k/2} [f_{x,\, k-2j,\, k}^{\,c} \cos(k - 2j)\gamma - f_{x,\, k-2j,\, k}^{\,s} \sin(k - 2j)\gamma] \tag{3.5}$$

$$\begin{aligned} F_r(\alpha, \gamma, \theta) = \sum_{k=0}^{n} \sin^k\alpha \sum_{0 \leqslant j \leqslant k/2} \{[f_{y,\, k-2j,\, k}^{\,c} \cos(k - 2j)\gamma - f_{y,\, k-2j,\, k}^{\,s} \sin(k - 2j)\gamma] \times \\ \times \cos(\theta - \gamma) + [f_{z,\, k-2j,\, k}^{\,c} \cos(k - 2j)\gamma - f_{z,\, k-2j,\, k}^{\,s} \sin(k - 2j)\gamma] \sin(\theta - \gamma)\} \end{aligned} \tag{3.6}$$

(For the sake of brevity, the arguments which are equal to zero have been dropped in (3.5) and (3.6).)

By virtue of (3.2) for $\theta = 0$ and $\theta = \pi/2$, and some rearranging, we obtain from (3.6) the expressions for $F_y(\alpha, \gamma)$ and $F_z(\alpha, \gamma)$:

$$F_y(\alpha, \gamma) = \sum_{k=1}^{n+1} \sin^{k-1}\alpha \sum_{0 \leqslant j \leqslant k/2} [f_{y\, k-2j,k}^{\prime c} \cos(k - 2j)\gamma - f_{y,k-2j,k}^{\prime s} \sin(k - 2j)\gamma] \tag{3.7}$$

$$F_z(\alpha, \gamma) = \sum_{k=1}^{n+1} \sin^{k-1}\alpha \sum_{0 \leqslant j \leqslant k/2} [f_{z,k-2j,k}^{\prime c} \cos(k - 2j)\gamma - f_{z,k-2j,k}^{\prime s} \sin(k - 2j)\gamma] \tag{3.8}$$

The coefficients with index k in (3.7) and (3.8) and the coefficients with index $k - 1$ in (3.6) are related one to one, and there exists the relation

$$f'^{c}_{ykk} = -f'^{s}_{zkk}, \qquad f'^{s}_{ykk} = f'^{c}_{zkk} \tag{3.9}$$

Hereafter, the primes in the coefficients of (3.7) and (3.8) will be dropped.

Equations (3.5), (3.7) and (3.8), together with the relations (3.9), determine the dependence of the components of **R** or **M** on the orientation of the body in the flow. Let us write these in a more compact form:

$$F_x(\alpha, \gamma) = \sum_{k=0}^{n} F_{xk}(\gamma)\sin^k\alpha, \qquad F_v(\alpha, \gamma) = \sum_{k=1}^{n+1} F_{vk}(\gamma)\sin^{k-1}\alpha \tag{3.10}$$

Here

$$F_{uk}(\gamma) = \sum_{0\leqslant j\leqslant k/2} [f^c_{u,k-2j,k}\cos(k-2j)\gamma - f^s_{u,k-2j,k}\sin(k-2j)\gamma] \tag{3.11}$$

and the index v stands for either y or z, while the index u stands for x or v.

Let us examine how Expressions (3.11) simplify in the case of a symmetric body. Let the body possess a plane of symmetry and that, for $\gamma = 0$, this plane coincides with the plane of the angle of attack $\beta = 0$. The flow field around the body (which is now rotated by an angle γ or $-\gamma$ from the original orientation) will still be symmetric with respect to the plane $\beta = 0$. Hence

$$\begin{aligned} &R_x(\gamma) = R_x(-\gamma), \qquad R_y(\gamma) = R_y(-\gamma), \qquad R_z(\gamma) = -R_z(-\gamma) \\ &M_x(\gamma) = -M_x(-\gamma), \quad M_y(\gamma) = -M_y(-\gamma), \quad M_z(\gamma) = M_z(-\gamma) \end{aligned} \tag{3.12}$$

and for R_x, R_y, M_z

$$f^s_{u,k-2j,k} = 0, \qquad F_{uk}(\gamma) = \sum_{0\leqslant j\leqslant k/2} f^c_{u,k-2j,k}\cos(k-2j)\gamma \tag{3.13}$$

while for R_z, M_x, M_y

$$f^c_{u,k-2j,k} = 0, \qquad F_{uk}(\gamma) = -\sum_{0\leqslant j\leqslant k/2} f^s_{u,k-2j,k}\sin(k-2j)\gamma \tag{3.14}$$

Further, let the body be symmetric in such a way that a rotation of the body by an angle $\gamma = 2\pi/m$ (m = integer > 1) around the x-axis cannot be distinguished from the original orientation. A body of revolution with regularly spaced and equally deflected control surfaces, for instance, possesses such symmetry.

From such "symmetry under finite rotation" it follows that

$$F_{uk}(\gamma + 2\pi/m) - F_{uk}(\gamma) = 0 \qquad (3.15)$$

Substituting (3.11) into (3.15) and simplifying, we obtain

$$\sum_{0\leqslant j\leqslant k/2}\Big[f_{u,k-2j,k}^{c}\sin(k-2j)\frac{\pi}{m}\sin(k-2j)\Big(\gamma+\frac{\pi}{m}\Big)+ + f_{u,k-2j,k}^{s}\sin(k-2j)\frac{\pi}{m}\cos(k-2j)\Big(\gamma+\frac{\pi}{m}\Big)\Big]=0 \qquad (3.16)$$

Because of the arbitrariness in the choice of the angle $\gamma + \pi/m$ it follows from (3.16) that

$$f_{u,k-2j,k}^{c}\sin(k-2j)\frac{\pi}{m}=0, \qquad f_{u,k-2j,k}^{s}\sin(k-2j)\frac{\pi}{m}=0 \qquad (0\leqslant j\leqslant k/2) \qquad (3.17)$$

Clearly $f^{c}_{u,k-2j,k}$ and $f^{s}_{u,k-2j,k}$ can be unequal to zero only for $k - 2j = im$, where i is an integer which satisfies $0 \leqslant i \leqslant k/m$ and which makes $k - im$ an even number.

In Equation (3.11) let us change from summing on j to summing on i, and let us introduce the function $\delta(i)$, which is zero for i odd and unity for i even. Then (3.11) is expressible as

$$F_{uk}(\gamma)=\sum_{0\leqslant i\leqslant k/m}\delta(k-\mathrm{im})(f_{u,\mathrm{im},k}^{c}\cos\mathrm{im}\,\gamma-f_{u,\mathrm{im},k}^{s}\sin\mathrm{im}\,\gamma) \qquad (3.18)$$

From (3.18) it can be seen that the smallest value of k for which $F_{uk}(\gamma)$ still depends on γ is $k = m$. It follows that for small angles of attack one can neglect the dependence of the aerodynamic forces on the orientation of the control surfaces relative to the plane of the angle of attack: in accordance with (3.10) the dependence of F_x on γ appears only in terms of order α^m, and for F_y and F_z in terms of order α^{m-1}.

For the case of an axisymmetric body, the number m can be considered as infinite, and for an arbitrary k there remains only one term in (3.18) corresponding to $i = 0$:

$$F_{uk} = \delta(k) f_{uk} \qquad (3.19)$$

(where instead of $f^{c}_{u,0,k}$ the notation f_{uk} is introduced).

Substituting (3.19) into (3.10), we obtain the expansions in powers of sin α of the aerodynamic forces on an axisymmetric body:

$$F_x(\alpha) = \sum_{k=0}^{n} \delta(k) f_{xk} \sin^k \alpha, \qquad F_v(\alpha) = \sum_{k=1}^{n+1} \delta(k) f_{vk} \sin^{k-1} \alpha \tag{3.20}$$

Since an axisymmetric body possesses a plane of symmetry, then from (3.12) and (3.20) it follows that $R_z = M_x = M_y = 0$.

4. As the free-stream velocity becomes more and more supersonic, the regions on the surface of the body which are influenced by the presence of the control surfaces become more and more narrow. Let us consider a velocity so large that for sin $\alpha \leqslant M$ and for all values of γ the regions of influence of the different control surfaces do not intersect on the surface of the body, i.e. there is no interference between the control surfaces.

Let Expressions (3.20) specify the magnitude of the aerodynamic forces and moments of an arbitrary axisymmetric body, and let (3.10) give the same quantities but for the same body with a single control surface. The difference between these two expressions $\Delta_1 F_u(\alpha, \gamma)$ gives the increments in forces and moments due to the effect of one control surface

$$\Delta_1 F_x(\alpha, \gamma) = \sum_{k=0}^{n} \Delta_1 F_{xk}(\gamma) \sin^k \alpha, \qquad \Delta_1 F_v(\alpha, \gamma) = \sum_{k=1}^{n+1} \Delta_1 F_{vk}(\gamma) \sin^{k-1} \alpha \tag{4.1}$$

where

$$\Delta_1 F_{uk}(\gamma) = -\delta(k) f_{uk} + \sum_{0 \leqslant j \leqslant k/2} [f^c_{u,\, k-2j,\, k} \cos(k-2j)\gamma - f^s_{u,\, k-2j,\, k} \sin(k-2j)\gamma] \tag{4.2}$$

When there are m regularly spaced identical control surfaces, then the total increments in forces and moments $\Delta_m F_u(\alpha, \gamma)$ are expressible in terms of $\Delta_1 F_u(\alpha, \gamma)$ (in view of the non-interference between surfaces):

$$\Delta_m F_u(\alpha, \gamma) = \sum_{l=0}^{m-1} \Delta_1 F_u\left(\alpha, \gamma + \frac{2\pi l}{m}\right) \tag{4.3}$$

Substituting (4.1) into (4.3)

$$\Delta_m F_x(\alpha, \gamma) = \sum_{k=0}^{n} \Delta_m F_{xk}(\gamma) \sin^k \alpha, \qquad \Delta_m F_v(\alpha, \gamma) = \sum_{k=1}^{n+1} \Delta_m F_{vk}(\gamma) \sin^{k-1} \alpha \tag{4.4}$$

where

$$\Delta_m F_{uk}(\gamma) = \sum_{l=0}^{m-1} \Delta_1 F_{uk}\left(\gamma + \frac{2\pi l}{m}\right) \tag{4.5}$$

Expanding the right-hand side of (4.5) with the aid of (4.2)

$$\Delta_m F_{uk}(\gamma) = -m\delta(k) f_{uk} + \sum_{0\leqslant j\leqslant k/2} (f_{u,\,k-2j,\,k}^{\,c}\,\sigma_{k-2j,m}^{\,c} - f_{u,\,k-2j,\,k}^{\,s}\,\sigma_{k-2j,m}^{\,s}) \qquad (4.6)$$

Here the following notation is used:

$$\sigma_{k-2j,m}^{\,c} = \sum_{l=0}^{m-1} \cos(k-2j)\left(\gamma + \frac{2\pi l}{m}\right), \qquad \sigma_{k-2j,\,m}^{\,s} = \sum_{l=0}^{m-1} \sin(k-2j)\left(\gamma + \frac{2\pi l}{m}\right) \qquad (4.7)$$

It is known (for instance [1]) that when the integer $k - 2j$ is not divisible by m, the sums are identically zero. When $k - 2j = im$, where i is an integer, then

$$\sigma_{im,m}^{\,c} = m\cos im\gamma, \qquad \sigma_{im,m}^{\,s} = m\sin im\gamma \qquad (4.8)$$

Let us substitute into (4.6) the values (4.7), and in (4.6) let us switch to summation on i as in the development of (3.18):

$$\Delta_m F_{uk}(\gamma) = m\left[-\delta(k) f_{uk} + \sum_{0\leqslant i\leqslant k/m} \delta(k-im)\,(f_{u,\,im,\,k}^{\,c}\cos im\gamma - f_{u,\,im,\,k}^{\,s}\sin im\gamma)\right] \qquad (4.9)$$

Expressions (4.4) and (4.9) give the desired dependence of the aerodynamic forces and moments on the number of control surfaces m in absence of mutual interference. For $k < m$ the expression inside the brackets in (4.9) is independent of m. It follows that if we compare the contributions to forces or moments due to m and m' (with $m' > m$) control surfaces, we shall have

$$\frac{\Delta_{m'} F_x}{\Delta_m F_x} = \frac{m'}{m}, \qquad \frac{\Delta_{m'} F_{y'}}{\Delta_m F_y} = \frac{\Delta_{m'} F_z}{\Delta_m F_z} = \frac{m'}{m} \qquad (4.10)$$

The first equality is valid to order α^m and the second to order α^{m-1}.

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